Producto de una serie cualqueira por la exponencial e^{-x}. (Estrait du Progreso Matemático de Zaragoza, ser. 2'. An. II, n°. 10, 1900).

Una trasformacion del Prof. Allardice (extr. ibid. n°. 11) (1900).

Alcune combinazioni di formule (extr. ibid. n°. 17) (1901).

Su di alcune proprieta delle Superfici a generatrice circolare (Supplem. al Vol. 1 delle Matematiche).

Una costruzione geometrica dell'equazione cubica (nel Vol. 1, n°. 5 del "Le Matematiche").

Une méthode élémentaire de recherche del maxima et minima. (Extr. de la Gazeta Matematica. An. VII, n°. 9, Mai 1902, Buçuresti,—(Rumania).

Saggio di nomenclatura della Recente Geometria del triangolo (dans le journal "Il Pitagora," vol. VIII, n°. 3a9, 1902).

Alcune asservazioni sui pendoli e sui cronometri (Extr. de la Rassegua Tecnica Italiana, An. II, n°. 4e5, Messina, 1902).

Complementi di Geometria elementare,—Milano, 1902, U. Hoepli, ed.

Elementi di Trigonometria Piana e Sferica, traduction, avec notej et adjointes du Traité de M. l'Abbé H. Gelin, de Huy.

Trattáto d Aritmetica, en collaboration de M. Gelin, à Huy.

Alcune formule della Teoria delle Superficie (Extr. de la Revista simestral de Matemática, An. II, n°. 6, Zaragoza, 1902).

NINTH SUMMER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.*

By DR. L. E. DICKSON, of the University of Chicago.

The appropriateness of selecting a place as far west as Evanston, Illinois, for one of the Summer meetings of the whole Society was shown by the large and representative attendance as well as by the enthusiasm evinced at the four official sessions and at the several social concourses. Bearing on the geography of the subject is the fact that Chicago contributes the President of the Society, Professor E. H. Moore. There were present members from Columbia, Cornell, Johns Hopkins and other eastern institutions; from Kansas, the Dakotas, and other western States; while the middle west was very fully represented. The number of papers presented exceeded thirty, being equal to the number presented last Summer at Ithaca.

The program opened Tuesday morning, September second, with a paper by Dr. F. R. Moulton of the University of Chicago, entitled "A method of constructing general expressions for the elements of the planetary orbits which are valid for a finite time," in which objection was made to the so-called proofs by

*This report was written by request of the Editor.

the French astronomers of the past century of the stability of the solar system for infinite time.

Under certain restrictions, Dr. Moulton established rigorously the existence of periodic solutions. The important work of Dr. Moulton in introducing modern mathematical rigor in astronomical investigations cannot be too highly commended.

Professor Hathaway of Rose Polytechnic Institure, read a paper on "The quaternion treatment of the problem of three bodies," obtaining very condensed expressions for known results. In the discussion, several members expressed doubt as to the power of quaternion analysis for the discovery of new results.

Professor Dickson of the University of Chicago presented "Definitions of a linear associative algebra by independent postulates," as well as "Two definitions of a field by independent postulates;" giving also an account of the related paper by Dr. E. V. Huntington of Harvard University. Several members participated in a discussion of these three papers and the more genersl topic of axioms and postulates, as also illustrated by Hilbert's work in Geometry and the group definitions by Huntington and Moore. One point emphasized was the advantage of simple natural postulates, even if the *number* of independent postulates was larger than for a more artificial system. The three papers will appear in the *Transactions* of the Society.

The first paper on Tuesday afternoon was by Professor Oskar Bolza of the University of Chicago, who gave "Some instructive examples in the calculus of variations." To indicate the pedagogical value of the paper, it may be remarked that a volume by Professor Bolza on the Calculus of Variations is soon to appear in the Decennial Publications of the University of Chicago.

Professor J. B. Shaw of Kenyon College, read "On linear associative algebras," the paper being a continuation of the one read recently before the Chicago Section.

Dr. W. B. Fite of Cornell University, read "Concerning the commutator subgroups of groups whose orders are powers of primes."

Mr. J. W. Young of Cornell, presented a very clear preliminary report "On a certain group of linear substitutions and the functions belonging to it." The paper has close contact with Klein's Elliptic Modular Functions.

A paper on the topic "In technical theory of numbers," by Professor Westlund of Purdue University, was presented in abstract by Professor Moore.

Professor L. E. Dickson of Chicago, made a brief "Announcement of new simple groups," related to his paper in the *Transactions*, July, 1901.

On Wednesday morning, Professor Maschke gave a report on the paper "Ueber die Reducibilität der Gruppen linearer homogener Substitutionen" by Professor Alfred Loewy of the University of Freiburg. The paper will appear shortly in the *Transactions*.

Professor H. S. White of Northwestern University presented "A special twisted cubic with rectilinear directrix," generalizing certain interesting properties of conics.

Professor F. Morley of Johns Hopkins University applied the geometry of the complex variable to obtain "Orthocentric properties of the plane n-line." He announced theorems relating to a general system of straight lines.

Four other papers on the programme and read by title were of geometrical character; they were by Dr. Snyder of Cornell, Professor Field of Carthage College, Dr. Keyser of Columbia University, Professor Emch of Colorado University.

On Wednesday afternoon were presented several papers not on the printed programme, also (by title) papers by Dr. Eisenhart of Princeton, Professor Roe of Syracuse University, Dr. Epsteen of Philadelphia, Mr. Ford of the University of Michigan, and finally the following three papers:

"The bilinear point-line connex in space; an extension of Clebsch's connex," by Dr. E. Kasner of Columbia University; "Multiple points of Lissajous's curves in two and three dimensions," by Mr. E. A. Hook, Milwaukee; "Null systems in space of five dimensions," by Professor John Eiesland, Theil College.

A large number of the members lunched together each day at Evanston's one hotel, and dined together in one of the College buildings Tuesday evening, afterwards visiting the University observatory. On Thursday, the day after the Society adjourned, a number of members visisted the Chicago Steel Works and the University of Chicago, lunching together at the Quadrangle Club.

THE EXPRESSION OF THE nth POWER OF A NUMBER IN TERMS OF THE nth POWERS OF OTHER NUMBERS, n BEING ANY INTEGER; AND THE DEDUCTION OF SOME INTERESTING PROPERTIES OF PRIME NUMBERS.*

By J. W. NICHOLSON, A. M., LL. D., Professor of Mathematics in the Louisiana State University.

Any polynomial of m terms is equal to, and may be expressed under the form of:

$$
\begin{aligned}
&+[\text{The sum of its terms taken in sets of } m-1]\\
&-[\ \text{“}\quad\text{“}\quad\text{“}\qquad\text{“}\quad\text{“}\quad m-2]\\
&+[\ \text{“}\quad\text{“}\quad\text{“}\qquad\text{“}\quad\text{“}\quad m-3]\\
&-[\ \text{“}\quad\text{“}\quad\text{“}\qquad\text{“}\quad\text{“}\quad m-4]\\
&\quad\ldots\ldots\ldots\ldots
\end{aligned}
$$

*This paper was read to the American Mathematical Society several years ago and in presenting it the author asked whether or not anything of the kind had ever before been discovered. The inquiry was referred to Dr. Artemas Martin who reported that the main formula, designated in this paper by (4), had been discovered and presented to the London Mathematical Society, but that the paper contained none of the transformations or deductions embodied in my discovery. With the hope that others will be interested in the matter and reach still more beautiful and important deductions, the paper as presented by me to the Society is now submitted to the public.

EXAMPLES.

1. $a_1+a_2+a_3=[(a_1+a_2)+(a_1+a_3)+(a_2+a_3)]-[(a_1)+(a_2)+(a_3)]$....(1).

2. $a_1+a_2+a_3+a_4=$

$$\left.\begin{array}{l} +[(a_1+a_2+a_3)+(a_1+a_2+a_4)+(a_1+a_3+a_4)+(a_2+a_3+a_4)] \\ -[(a_1+a_2)+(a_1+a_3)+(a_1+a_4)+(a_2+a_3)+(a_2+a_4)+(a_3+a_4)] \\ +[(a_1)+(a_2)+(a_3)+(a_4)] \end{array}\right\}\text{....(2).}$$

3. In general, making $s=a_1+a_2+....a_m$, we have

$$\left.\begin{array}{rl} s=[(s-a_1)+(s-a_2)+ & .\quad .\quad .\quad (s-a_m)] \\ -[s-a_1-a_2)+(s-a_1-a_3)+ & .\quad (s-a_{m-1}-a_m] \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot & \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ -(-1)^m[(a_1+a_2)+(a_1+a_3)+ & .\quad .\quad .\ (a_{m-1}+a_m)] \\ (-1)^m[(a_1)+(a_2)+....(a_m)] & \end{array}\right\}\text{....(3).}$$

That (3) is true may be shown thus:

Any term of s, as a_1, occurs in the first [ ] $m-1$ times; in the second [ ], $\frac{(m-1)(m-2)}{2}$ times; in the third, $\frac{(m-1)(m-2)(m-3)}{2.3}$ times; and so on to the last in which it occurs 1 time. Hence the sum of all the a_1's in the second member is

$$[(m-1)-\frac{(m-1)(m-2)}{2}+....(-1)^m]a_1=a_1. \qquad \text{Q. E. D.}$$

In all that follows, n is supposed to be a positive integer and less than m, unless otherwise stated.

An interesting and important property of polynomials arranged under the form of (3) is set forth in the following

PROPOSITION: *The second member of (3) may be raised to the nth power by raising each of the parts within the ( ) to that power, ignoring the signs before the* [ ]. Thus:

$$\left.\begin{array}{rl} s^n=[(s-a_1)^n+(s-a_2)^n+ & .\quad .\quad .\quad (s-a_m)^n] \\ -[(s-a_1-a_2)^n+(s-a_1-a_3)^n+ & .\quad (s-a_{m-1}-a_m)^n] \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot & \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ -(-1)^m[(a_1+a_2)^n+(a_1+a_3)^n+ & .\quad .\quad (a_{m-1}+a_m)^n] \\ (-1)^m[(a_1)^m+(a_2)^n+....(a_m)^n] & \end{array}\right\}\text{....(4).}$$

PROOF. Let us suppose s^n, and each of the terms of the second member, to be expanded. Let $G=m'a_1^p a_2^q....a_k^z$ be any particular term of s^n, where $p+q+....z=n$, and k any integer from 1 to m. Now G occurs $m-k$ times in the

first [ ], $\frac{(m-k)(m-k-1)}{2}$ times in the second [ ], $\frac{(m-k)(m-k-1(m-k-2)}{2.3}$ times in the third [ ], and so on. Hence the sum of all the G's in the second member is

$$[m-k-\frac{(m-k)(m-k-1)}{2}+....(-1)^m]G=G, \qquad \text{Q. E. D.}$$

EXAMPLES.

1. Making $a_1=2$, $a_2=3$, $a_m=a_3=6$, whence $s=11$, and we have

$$11^n=9^n+8^n+5^n-6^n-3^n-2^n....(5),$$

where $n=2$ or 1.

2. Making $a_1=2$, $a_2=3$, $a_3=6$, $a_m=a_4=12$, we have

$$23^n=21^n+20^n+17^n+11^n-18^n-15^n-14^n-9^n-8^n-5^n+12^n+6^n+3^n+2^n....(6),$$

where $n=3$, 2 or 1.

3. Making $a_1=1$, $a_2=3$, $a_3=7$, $a_4=15$, $a_m=a_5=31$, we have

$$57^n=56^n+54^n-53^n+50^n-49^n-47^n+46^n+42^n-41^n-39^n+38^n-35^n+34^n+32^n$$
$$-31^n+26^n-25^n-23^n+22^n-19^n+18^n+16^n-15^n-11^n+10^n+8^n-7^n+4^n$$
$$-3^n-1^n....(7).$$

Here $n=4$, 3, 2 or 1.

Some additional properties of an interesting character are presented in the following corollaries to the preceding proposition.

COR. I. If $s=2^m-1$, then s^m may be expressed in terms of the nth powers of all the integers from 2^m-2 to 1, inclusive.

EXAMPLES.

1. For $a_1=1$, $a_2=2$, $a_m=a_3=4$, $s=7=2^3-1$, we have

$$(2^3-1)^n+5^n-4^n+3^n-2^n-1^n....(8),$$

where $n=2$ or 1.

2. For $a_1=1$, $a_2=2$, $a_3=4$, $a_m=a_4=8$, $s=15=2^4-1$, we have

$$(2^4-1)^n=14^n+13^n-12^n+11^n-10^n-9^n+8^n+7^n-6^n-5^n+4^n$$
$$-3^n+2^n+1^n....(9),$$

where $n=3$, 2 or 1.

NOTE.—By (8) we observe that the sum of the last seven terms of (9)=0. Hence,

$$15^n = 14^n + 13^n - 12^n + 11^n - 10^n - 9^n + 8^n \ldots\ldots (10),$$

where $n=2$ or 1.

3. In general, to find $(2^m-1)^n$ we make $a_1=1$, $a_2=2$, $a_3=2^2$, $a_4=2^3$,... $a_m=2^{m-1}$, and obtain

$$(2^m-1)^n = (2^m-2)^n + (2^m-3)^n - (2^m-4)^n \ldots\ldots - (-1)^m 3^n + (-1)^m 2^n + (-1)^m \ldots\ldots (11),$$

where $n=m-1$, $m-2$, ...2, or 1.

Cor. II. If $s=(m-1)a_2 + \left(\frac{(m-1)(m-2)}{2}+1\right)a_1$, then s^n may be expressed in terms of the nth powers of $2^m-2-2(m-2)^2$ integers, or less.

This is done by making $a_3=a_2+a_1$, $a_4=a_3+a_1$, $a_5=a_4+a_1$, etc., in (4), and eliminating the equal terms having contrary signs.

Examples.

1. Make $m=3$, $a_3=a_2+a_1$ and (4) becomes

$$(2a_2+2a_1)^n = (2a_2+a_1)^n + (a_2+2a_1)^n - a_2^n - (a_1)^n \ldots\ldots (12),$$

where $n=2$ or 1.

Thus, for $a_2=7$, $a_1=5$, and $n=2$, we have

$$24^2 = 19^2 + 17^2 - 7^2 - 5^2 \ldots\ldots (13).$$

2. Make $m=4$, $a_4=a_3+a_1$, $a_3=a_2+a_1$, and we have

$$(3a_2+4a_1)^n = (3a_2+3a_1)^n + (2a_2+4a_1)^n - (a_2+3a_1)^n - (2a_2+a_1)^n + a_2^n + a_1^n \ldots\ldots (14),$$

where $n=3$, 2, or 1.

Thus, for $a_2=7$, $a_1=5$, $n=3$, we have

$$41^3 = 36^3 + 34^3 - 22^3 - 19^3 + 7^3 + 5^3 \ldots\ldots (15).$$

3. Make $m=5$, $a_5=a_4+a_1$, $a_4=a_3+a_1$, $a_3=a_2+1$, and we have

$$(4a_2+7a_1)^n = (4a_2+6a_1)^n + (3a_2+7a_1)^n - (2a_2+6a_1)^n - (3a_2+3a_1)^n - (2a_2+4a_1)^n + (2a_2+3a_1)^n + (a_2+4a_1)^n + (2a_2+a_1)^n - a_2^n - a_1^n \ldots (16),$$

where $n=4$, 3, 2, or 1.

4. Making $m=6$, $a_6=a_5+a_1$, $a_5=a_4+a_1$, etc., we have

$$(5a_2+11a_1)^n = (5a_2+10a_1)^n + (4a_2+11a_1)^n - (4a_2+6a_1)^n - (3a_2+10a_1)^n - (3a_2+8a_1)^n + (3a_2+5a_1)^n + (2a_2+8a_1)^n + (3a_2+3a_1)^n$$

$$+(2a_2+6a_1)^n-(a_2+5a_1)^n-(2a_2+3a_1)^n-(2a_2+a_1)^n$$
$$=a_2{}^n+a_1{}^n \ldots\ldots(17),$$

where n=5, 4, 3, 2, or 1.

Thus, making a_2=5, a_1=2, we have

$$47^n=45^n+42^n-35^n-32^n-31^n+26^n+25^n+2^n+5^n-12^n$$
$$-15^n-16^n+21^n+22^n \ldots\ldots(18),$$

where n=5, 4, 3, 2, or 1.

Making a_2=13, a_1=−5, transposing, we have

$$24^n=22^n+21^n-15^n-13^n-12^n+10^n+5^n+4^n+3^n-1^n \ldots\ldots(19),$$

where n=5, 3, or 1.

Making a_2=8, a_1=−3, transposing, we have

$$15^n=14^n+13^n-10^n-9^n+7^n-6^n+3^n+2^n+1^n \ldots\ldots(20),$$

where n=5, 3, or 1.

Cor. III. The root of each term in (4) may be increased by any number, as c, provided that the second member be increased by $-(-1)^m c^n$. Thus:

$$\begin{aligned}(s+c)^n=&+[(s+c-a_1)^n+(s+c-a_2)^n+\ .\ .\ .\ (s+c-a_m)^n]\\&-[(s+c-a_1-a_2)^n+(s+c-a_1-a_3)^n+\ldots.(s+c-a_{m-1}-a_m)^n]\\&.\ .\ .\ .

In (21), by making $a_1=a_2=....a_m$, we have

$$(c+ma_1)^n=m[c+(m-1)a_1]^n-\frac{m(m-1)}{2}[c+(m-2)a_1]^n+..-(-1)^m c^n \ldots(25).$$

By (25) we may express the nth power of $c+ma_1$ in terms of the nth powers of the terms of the arithmetical progression

$$c+(m-1)a_1,\quad c+(m-2)a_1,\quad c+(m-3)a_1, \text{ etc.},$$

multiplied by the binomial coefficients, m, $\frac{m(m-1)}{2}$, etc., respectively. Thus:

Making $c=5$, $m=6$, $a_1=7$, we have

$$47^n=6.40^n-15.33^n+20.26^n-15.19^n+6.12^n-5^n \ldots(26),$$

where $n=5$, 4, 3, 2, or 1.

Making $c=41$, $m=6$, $a_1=1$, we have

$$47^n=6.46^n-15.45^n+20.44^n-15.43^n+6.42^n-41^n \ldots(27).$$

Making $c=-15$, $m=6$, $a_1=6$, we have

$$21^n=6.15^n-15.9^n+20.3^n-15.(-3)^n+6(-9)^n-(-15)^n \ldots(28).$$

Hence, $21^n=5.15^n-9.9^n+5.3^n$, where $n=4$ or 2; and $21^n=7.15^n-21.9^n+35.3^n$, where $n=5$, 3, or 1.

Making $c=0$ and $a_1=1$, we have

$$m^n=m(m-1)^m-\frac{m(m-1)}{2}(m-2)^n+\ldots.+(-1)^m m.1^n \ldots(29).$$

Thus, for $m=6$, we have $6^n=6.5^n-15.4^n+20.3^n-15.2^n+6$, where $n=5$, 4, 3, 2, or 1.

Cor. IV. Multiplying (4) by 2^n, and then adding $c-s$ to the root of each term, we have

$$\begin{aligned}(c+s)^n=&[(c+s-2a_1)^n+(c+s-2a_2)^n+\quad .\quad .\quad (c+s-2a_m)^n]\\&-[(c+s-2a_1-2a_2)^n\quad .\quad .\quad .\quad (c+s-2a_{m-1}-2a_m)^n]\\&\quad .\quad .\quad .

EXAMPLES.

Making $a_1=1$, $a_2=2$, $a_3=3$, $a_m=a_4=5$, $c=7$, we have

$$18^n=16^n+14^n-10^n-2^n, \text{ or } 9^n=8^n+7^n-5^n-1^n \ldots.(31),$$

where $n=3$ or 1.

Making $a_1=1$, $a_2=2$, $a_3=3$, $a_4=5$, $a_5=8$, $a_6=a_m=13$, we have

$$\begin{aligned}(c+32)^n=&(c+30)^n+(c+28)^n-(c+24)^n-(c+18)^n+(c+16)^n-(c-10)^n\\&+(c+8)^n+(c+6)^n-(c+4)^n-(c-4)^n+(c-6)^n+(c-8)^n \quad ..(32),\\&-(c-10)^n+(c-16)^n-(c-18)^n-(c-24)^n+(c-28)^n+(c-30)^n\\&-(c-32)^n\end{aligned}$$

where $n=5$, 4, 3, 2, or 1.

Making c, in (32), $=7$, we have

$$39^n=37^n+35^n-31^n+15^n+13^n-9^n-21^n \ldots.(33),$$

where $n=5$, 3, or 1.

In (30), by making $c=a_1=a_2=\ldots.a_m=1$, we have

$$(m-1)^n=m(m-1)^n-\frac{m(m-1)}{2}(m-3)^n+\frac{m(m-1)(m-2)}{6}(m-5)^n-\ldots.$$

$$\ldots.-(-1)^m\frac{m(m-1)}{2}(5-m)^n+(-1)^m m(3-m)^n-(-1)^m(1-m)^n \ldots.(34).$$

Supposing m and n to be both even or both odd, (34) may be written

$$(m+1)^n=(m-1)^{n+1}-\frac{m}{2}(m-3)^{n+1}+\frac{m(m-1)}{6}(m-5)^{n+1}$$

$$-\frac{m(m-1)(m-2)}{24}(m-7)^{n+1}+\ldots(35).$$

In (35), when m is even, the root of the last term is 1, and when the former is odd the latter is 2. Thus:

Making $m=6$, we have

$$7^n=5^{n+1}-3.3^{n+1}+5.1^{n+1}\ldots.(36),$$

where $n=4$ or 2.

[To be Continued.]

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ARITHMETIC.

159. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.

The amount of tax assessed on the property of a city is T,=\$145,850; and the treasurer was allowed a fee of $m\%$,=$\frac{3}{4}\%$, for collection. If $n\%$,=10%, of the tax was uncollectable, what were the net proceeds of the tax?

Solution by G. B. M. ZERR, A. M., Ph. D., The Temple College, Philadelphia, Pa., and J. R. HITT, Coronal Institute, San Marcos, Tex.

$$\text{Net proceeds} = T(100\% - m\%)(100\% - n\%)$$
$$= \$145{,}850 \times .90 \times .99\tfrac{1}{4} = \$130{,}280.51\tfrac{1}{4}.$$

If he was allowed $m\% = \frac{3}{4}\%$ of the whole, then net proceeds
$$= T(100\% - m\% - n\%)$$
$$= \$145{,}850 \times .89\tfrac{1}{4} = \$130{,}171.12\tfrac{1}{2}.$$

160. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.

A farm is rented for \$$R$,=\$300, in cash and a certain number of bushels of wheat. When wheat is \$$n$,=\$4-5 per bushel the rent is $p\%$,=$12\frac{1}{2}\%$ lower than when wheat is \$$m$, =\$1 1-5 per bushel. Find the number of bushels of wheat.

Solution by D. B. NORTHRUP, Mandana, N. Y.

Let x=the number of bushels of wheat. Then, by the conditions of the problem, $R+nx$=the number of dollars in the rent, when wheat is \$$n$ per bushel, and $R+mx$=the number of dollars in the rent when wheat is \$$m$ per bushel. But, also by the conditions of the problem, $R+nx=(1-p)(R+mx)$. Solving this equation for x, we find $x=\dfrac{pR}{(1-p)m-n}$. Substituting the numbers for the letters, x=150, the number of bushels of wheat.

ALGEBRA.

153. Proposed by JOHN M. COLAW, A. M., Monterey, Va.

Eliminate x, y, z, from the equations

$$x^2+yz=a,$$
$$y^2+zx=b,$$
$$z^2+xy=c,$$
$$x+y+z=0.$$

Solution by MARCUS BAKER, Washington, D. C.

Let $s=a+b+c$; then

$$s+a=y^2+2yz+z^2+x(2x+y+z)=(y+z)^2+x^2=2x^2;$$
similarly, $s+b=\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad=2y^2;$
$$s+c=\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad=2z^2.$$

Hence $\sqrt{(s+a)}+\sqrt{(s+b)}+\sqrt{(s+c)}=(x+y+z)\sqrt{2}=0$, an equation involving only a, b, c.

If we wish to express this relation without radicals we transpose, square, and reduce; whence $a-s=\sqrt{[(s+b)(s+c)]}$, whence

$$s^2+as=a^2-bc$$
$$s^2+bs=b^2-ca$$
$$s^2+cs=c^2-ab.$$

Whence $4s^2=a^2+b^2+c^2-(ab+bc+ca)$; and therefore

$$a^2+b^2+c^2+3(ab+bc+ca)=0.$$

This may also be written

$$\frac{a}{bc}+\frac{b}{ca}+\frac{c}{ab}+3\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)=0.$$

Example. Let $x=+1$, $y=+2$, $z=-3$; then $a=-5$, $b=+1$, $c=+11$.

Excellent solutions of this problem were received from *F. L. SAWYER, LON C. WALKER, JOSIAH H. DRUMMOND, J. K. ELLWOOD, J. SCHEFFER*, and *G. B. M. ZERR*.

Mr. Baker sent in neat solutions of problems 151 and 152.

154. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.

Show that the equation, $x^4+qx^2+s=0$....(1), can not have three *equal* roots.

Solution by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy in Ohio University, Athens, Ohio.

By the usual theory, the conditions that $ax^4+4bx^3+6cx^2+4dx+e=0$..(1) shall have three equal roots are

$$ae-4bd+3c^2=0....(2),$$
$$ad^2+eb^2+c^3-ace-2bcd=0....(3).$$

In the given equation, $x^4+qx^2+s=0$....(4), $a=1$, $b=0$, $c=\frac{1}{6}q$, $d=0$, $e=s$, and (2) and (3) become

$$s+\frac{q^2}{12}=0....(5),\quad \frac{q^3}{36}-qs=0....(6).$$

Eliminating s from (5) and (6) gives $q=0$, which is inconsistent with the supposition.

Also solved similarly by *G. B. M. ZERR, F. L. SAWYER, JOSIAH H. DRUMMOMD, J. R. HITT,* and *HARRY S. VANDIVER.*

155. Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.

If the roots of the cubic $x^3+3px^2+3qx+r=0$ be in harmonical progression, $2q^3=r(3pq-r)$.

Solution by HOMER R. HIGLEY, M. S., State Normal School, East Stroudsburg, Pa.; J. R. HITT, Coronal Institute, San Marcos, Tex.; H. S. VANDIVER, Bala, Pa.; and the PROPOSER.

If α, β, γ be the roots of a cubic, we have

$$x^3-(\alpha+\beta+\gamma)x^2+(\alpha\beta+\alpha\gamma+\beta\gamma)x-\alpha\beta\gamma=0....(1).$$

By the condition of the problem,

$$\beta=\frac{2\alpha\gamma}{\alpha+\gamma}, \text{ or } \alpha\beta+\alpha\gamma+\beta\gamma=3\alpha\gamma....(2),$$

and (1) becomes

$$x^3-(\alpha+\beta+\gamma)x^2+3\alpha\gamma x-\alpha\beta\gamma=0....(3).$$

Comparing (3) and

$$x^3+3px^2+3qx+r=0....(4),$$

$$\alpha+\beta+\gamma=-3p....(5), \quad \alpha\gamma=q....(6), \quad \text{and } \alpha\beta\gamma=-r....(7).$$

From (6) and (7), $\beta=-r/q....(8)$.
From (5) and (8), $\alpha+\gamma=r/q-3p....(9)$.
Eliminating α, β, γ from (2) by use of (6), (8), (9), $2q^3=r(3pq-r)$.

Also solved by *J. SCHEFFER, LON C. WALKER, JOSIAH H. DRUMMOND, F. L. SAWYER,* and *G. B. M. ZERR.*

GEOMETRY.

183. Proposed by S. F. NORRIS, Professor of Mathematics and Astronomy in Baltimore City College, Baltimore, Md.

Two quadrilaterals having three sides of the one equal to the three corresponding sides of the other, each to each, and the two corresponding angles adjacent to the unknown sides equal, each to each, are equal figures. [*Olney's Geometry*, page 129.]

This is problem 169, an erroneous demonstration of which was published in the April number of the current volume of the MONTHLY. The fallacy in the demonstration is pointed out in Dr. Halsted's article, "Proving the False," on page 129. ED.

184. Proposed by ERWIN MARTIN, Principal of Schools, Mead, Neb.

If from any point in the circumference of a circle circumscribed about a triangle, perpendiculars are drawn to the sides, or the sides produced, of the inscribed triangle, the lines connecting the feet of the perpendiculars are collinear.

Demonstration by J. R. NITT, Coronal Institute, San Marcos, Tex.; G. I. HOPKINS, A.M., High School, Manchester, N. H.; and G. W. GREENWOOD, B. A., McKendree College, Lebanon, Ill.

Let ABC be a triangle; PD, PE, PF perpendiculars from any point P of circumcircle upon the sides.

To prove D, E, F collinear: Draw BP and CP, DE and DF. The quadrilaterals $BEPD$, $CFPD$ are cyclic.

$\therefore \angle PDE = \angle PBE$, $\angle PDF = \angle PCF$. But $\angle PBE = \angle PCF$, each being measured by $\frac{1}{2}$arcAP.

$\therefore \angle PDE = \angle PDF$. $\therefore$ D, E, F are collinear.

Also demonstrated by *BEULAH FRAZIER*, Soph., Rolla School of Mines, and *G. B. M. ZERR*.

162. Proposed by J. D. PALMER, Providence, Ky.

Given the distances from the vertices of a triangle, ABC, to the center of the incircle, to construct the triangle.

Solution by MARCUS BAKER, U. S. Geological Survey, Washington, D. C.

Already two solutions of this problem have appeared in the MONTHLY, an erroneous one in the January number, page 15, and a revised solution in the February number, pages 45-46, yet a word or two more may not be superfluous.

The problem is to *construct* a triangle knowing the distances from the incenter to the vertices.

The correct analysis of the problem, as Zerr shows in his revised solution results in a cubic equation, which means that the problem is insoluble, *i. e.* insoluble in the same sense that the trisection of an angle is impossible. The true answer is that the construction called for can not be made by elementry geometry.

The following is a trigonometrical analysis of the problem.

If A, B, C are the angles of a plane triangle, then

$$2\sin\tfrac{1}{2}A\sin\tfrac{1}{2}B\sin\tfrac{1}{2}C+\sin^2\tfrac{1}{2}A+\sin^2\tfrac{1}{2}B+\sin^2\tfrac{1}{2}C=1....(1).$$

This theorem readily results from substituting in the following well known relation

$$\cos A+\cos B+\cos C-1=4\sin\tfrac{1}{2}A\sin\tfrac{1}{2}B\sin\tfrac{1}{2}C$$

for $\cos A$, $1-2\sin^2\frac{1}{2}A$, etc., and reducing.

Now let α, β, γ be the distances from the incenter to the vertices and r the radius of the inscribed circle; whence

$$\sin\tfrac{1}{2}A=r/\alpha,\quad \sin\tfrac{1}{2}B=r/\beta,\quad \sin\tfrac{1}{2}C=r/\gamma.$$

Substituting in (1), we have

$$\frac{2r^3}{\alpha\beta\gamma}+\left(\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}\right)r^2=1, \text{ or } r^3+\tfrac{1}{2}\left(\frac{\alpha\beta}{\delta}+\frac{\beta\gamma}{\alpha}+\frac{\gamma\alpha}{\beta}\right)r^2=\tfrac{1}{2}\alpha\beta\gamma.$$

Example. Let $\alpha=275$, $\beta=325$, $\gamma=429$; then $r^3+539\frac{1}{6}r^2-19170937\frac{1}{2}=0$; whence $(r-165)(r^2+704\frac{1}{6}r+116187\frac{1}{2})=0$.

$\therefore r=165$. Also, $r=-137.97+$ and $r=842.13+$.

CALCULUS.

140. Proposed by C. C. BEBOUT, Professor of Mathematics, Elgin High School, Elgin, Ill.

A pole two inches in diameter is set vertically in a level plat of ground. At a point ten feet from the ground a string is attached. A man holds the other end of the string and walks about the pole keeping the string stretched taut, and his hand at a constant distance of four feet from the ground, till the string is all wound upon the pole. If string is ten feet long, how far has his hand moved in the operation?

Solution by G. B. M. ZERR, A.M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let n be the number of times the string 10 feet feet in length will encircle the cylinder in a height of 6 feet. Circumference of cylinder$=\frac{1}{6}\pi$ feet, $x=$distance between successive portions of the string on the same generating line.

Then $n\sqrt{[(\frac{1}{6}\pi)^2+x^2]}=10/n$ also $nx=6$.

$$\therefore n\sqrt{\left(\frac{\pi}{6}\right)^2+\left(\frac{6}{n}\right)^2}=10 \text{ or } \frac{\pi^2n^2}{36}=64. \quad \therefore n=\frac{48}{\pi},\ x=\tfrac{1}{8}\pi.$$

The distance moved by the hand is given on page 319, No. 9, Vol. I, of the MONTHLY, and is there worked out in full. It is $s=2\pi^2r^2m^2$ where r is the radius of the cylinder and m is the number of times the string is wound around the cylinder or the same as the value of n above. $r=\frac{1}{12}$ feet, $m=n=48/\pi$.

$\therefore s=32$ feet.

Also solved by *T. T. DAVIS*.

141. Proposed by B. F. FINKEL, A.M., M.Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

The curve $r^n=a^n\sin n\theta$ rolls along a straight line. Show that the intrinsic equation to the evolute of the locus of the pole is $s^n=a^m[1+1/n]^n\sin n\phi$. [Edward's *Differential Calculus*, page 502.

Solution by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.

The distance from the point of tangency to the pole is normal to the path of the pole. Let this distance, which is a radius vector of the rolling curve, be r, and the perpendicular from the pole upon the fixed line be p.

Taking the fixed line as the axis of x, p is an ordinate of the pole, and r is the length of normal and is expressed by

$$y\sqrt{1+\frac{dy^2}{dx^2}}.$$

Now, all curves of the form $r^m=a^m\cos m\theta$ have their pedal equations in the form $r^{m+1}=a^m p$....[a].

It is plain that the given curve may be so expressed. The equation [a] then gives for the differential equation of the locus of the pole

$$\left(y\sqrt{1+\frac{dy^2}{dx^2}}\right)^{n+1}=a^n y, \text{ or, } \frac{dy}{dx}=\sqrt{[a^{2n/(n+1)}-y^{2n/(n+1)}\div y^{n/(n+1)}},$$

in which $dy/dx=\cot\varphi$ from the form of the given curve.

Then $y=a\sin^{(n+1)/n}\phi$, $\frac{dy}{d\phi}=\frac{n+1}{n}a\sin^{1/n}\phi\cos\phi$....[$b$].

Also, $\frac{dx}{dy}=\tan\phi$, $\frac{ds}{dy}=\frac{1}{\cos\phi}$; then $\frac{ds}{d\phi}=\frac{n+1}{n}a\sin^{1/n}\phi$....[$c$].

But this gives the radius of curvature in the locus of the pole, which again is the length of arc corresponding in the evolute, which vanishes with ϕ.

$\therefore$ the evolute is $s=\frac{n+1}{n}a\sin^{1/n}\phi$, as required.

Also solved by *G. B. M. ZERR.*

142. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

Solve the differential equation,

$$[a-x]\frac{dz}{dx}+[b-y]\frac{dz}{dy}=c-z.$$

Solution by G. B. M. ZERR, A. M., Ph. D., The Temple College, Philadelphia, Pa.; WILLIAM HOOVER, A.M., Ph. D., Ohio University, Athens, O.; and the PROPOSER.

Putting, as usual, $dz/dx=p$, $dz/dy=q$, and then comparing the given equation with $Pp+Qq=R$, we have $P=a-x$, $Q=b-y$, $R=c-z$.

Then, $\frac{dx}{P}=\frac{dz}{R}$ gives $\frac{dx}{x-a}=\frac{dz}{z-c}$, and

$$\frac{dy}{Q}=\frac{dz}{R} \text{ gives } \frac{dy}{y-b}=\frac{dz}{z-c}.$$

These equations give $\frac{x-a}{z-c}=\alpha$, $\frac{y-b}{z-c}=\beta$; so that the solution is

$$\frac{b-y}{c-z}=\phi\left(\frac{a-x}{c-z}\right).$$

Solved similarly by *LON C. WALKER.*

143. Proposed by L. C. WALKER, A. M., Petaluma High School, Petaluma, Cal.

Find the area of greatest ellipse that can be inscribed in a given semicircle.

Solution by G. B. M. ZERR, A. M., Ph. D., The Temple College, Philadelphia, Pa.; J. SCHEFLER, A. M., Hagerstown, Md.; and the PROPOSER.

Let the equation of the ellipse be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1....(1),$$

and of the circle, $x^2+(y+b)^2=r^2....(2)$.

By eliminating x^2 from (1) and (2), then writing the condition for equal roots, we have

$$b^2=a^2-\frac{a^4}{r^2}....(3).$$

Required area$=\pi ab$, or say $u=a^2b^2=a^4-(a^6/r^2)$. Hence for a maximum, we get

$$4a^3=\frac{6a^5}{r^2}, \text{ or } a^2=\frac{2r^2}{3}. \quad \text{Then the area}=\frac{2\sqrt{3}}{9}\pi r^2.$$

Also solved by *JOSIAH H. DRUMMOND.*

MECHANICS.

138. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, Gloucestershire, England.

A smooth elliptical tube is held in the vertical plane with its major axis inclined to the vertical. A particle is projected from the lowest point. Find the pressure on the tube at any point and the condition that the pressure may vanish at the highest point.

Solution by G. B. M. ZERR, A. M., Ph.D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

The reaction on the tube is given by

$$v^2/\rho=R+X\rho(d^2x/ds^2)+Y\rho(d^2y/ds^2)=R+X(dy/ds)+Y(dx/ds).$$

Let the x-axis be vertical and let β be the inclination of the major axis to the vertical; then

$$(a^2\sin^2\beta+b^2\cos^2\beta)x^2+(a^2\cos^2\beta+b^2\sin^2\beta)y^2+2(a^2-b^2)xy\sin\beta\cos\beta=a^2b^2$$

or $Ax^2+By^2+2Cxy=a^2b^2$, is the equation to the tube.

But $x=r\cos\theta$, $y=r\sin\theta$.

$$\therefore r=\frac{ab}{\sqrt{(A\cos^2\theta+B\sin^2\theta+2C\sin\theta\cos\theta)}}$$

$$= \frac{ab\sqrt{2}}{\sqrt{[A+B+(A-B)\cos 2\theta + 2B\sin 2\theta]}} = \frac{ab\sqrt{2}}{D}.$$

$$dr = \frac{ab\sqrt{2}[(A-B)\sin 2\theta - 2C\cos 2\theta]d\theta}{D^3} = \frac{ab\sqrt{2}Ed\theta}{D^3}.$$

$$ds = \frac{2ab\sqrt{[A^2+B^2+2C^2+(A^2-B^2)\cos 2\theta + 2C(A+B)\sin 2\theta]}}{D^2} d\theta$$

$$= \frac{2abFd\theta}{D^3}, \quad dy = dr\sin\theta + r\cos\theta d\theta.$$

Now $X=-g$, $Y=0$. $\therefore R=v^2/\rho+g(dy/ds)$.

$$\frac{dy}{ds} = \frac{E\sin\theta + D^2\cos\theta}{F\sqrt{2}}. \quad \therefore R = \frac{v^2}{\rho} + \frac{g(E\sin\theta + D^2\cos\theta)}{F\sqrt{2}}.$$

When $R=0$ and $\theta=\pi$, $v^2 = \frac{Ag\rho}{\sqrt{(A^2+C^2)}} = \frac{g(Aa^2+Ab^2-a^2b^2)^{\frac{3}{2}}}{ab\sqrt{(A^3+AC^2)}}$.

If $\beta=0$, $v^2=b^2g/a$; if $a=b$, $v^2=ag$.

DIOPHANTINE ANALYSIS.

92. Proposed by M. A. GRUBER, A. M., War Department, Washington, D. C.

Find the sides of integral right triangles when the difference of the legs is given.

No solution of this problem has been received.

93. Proposed by the late SYLVESTER ROBBINS.

Solve and set forth twenty terms in some infinite series of rational parallelopipeds following the solid whose edges are 2, 3, 6, and diagonal 7.

Solution by G. B. M. ZERR, A.M., Ph.D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let x, $x+1$, x^2+x be the edges. Then $x^2+(x+1)^2+(x^2+x)^2=(x^2+x+1)^2$. $\therefore$ The diagonal$=x^2+x+1$.

No.	x	$x+1$	x^2+x	x^2+x+1
1	2	3	6	7
2	3	4	12	13
3	4	5	20	21
4	5	6	30	31
5	6	7	42	43
6	7	8	56	57
7	8	9	72	73

8	9	10	90	91
9	10	11	110	111
10	11	12	132	133
11	12	13	156	157
12	13	14	182	183
13	14	15	210	211
14	15	16	240	241
15	16	17	272	273
16	17	18	306	307
17	18	19	342	343
18	19	20	380	381
19	20	21	420	421
20	21	22	462	463
etc.	etc.	etc.	etc.	etc.

94. Proposed by L. C. WALKER, A. M., Petaluma High School, Petaluma, Cal.

Show that the area of a rational triangle cannot be a square number.

Solution by G. B. M. ZERR, A.M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let a^2+b^2, a^2+c^2, b^2+c^2 be the sides.

$\therefore$ Area$=abc\sqrt{(a^2+b^2+c^2)}$.

Let $a=m^2+mn$, $b=mn+n^2$, $c=mn$.

$\therefore$ Area$=mn(m^2+mn)(n^2+mn)(m^2+n^2+mn)$.

Let $n=1$. $\therefore$ Area$=m^2(m+1)^2(m^2+m+1)$.

For integral values of m, m^2+m+1 is not a square.

Let $9p$, $10p$, $17p$ be the sides. Then area$=36p^2$.

Let $3p$, $25p$, $26p$ be the sides. Then area$=36p^2$.

This gives two series of triangles whose areas are square numbers, thus proving that the proposition is not true.

95. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.

There are two unequal square numbers the sum of whose sum, difference, product, and quotient, is a square. Find the two numbers.

This is problem 91, so numbered by mistake. Its solution is in the April number, page 113. ED.

AVERAGE AND PROBABILITY.

115. Proposed by L. C. WALKER, A. M., Petaluma High School, Petaluma, Cal.

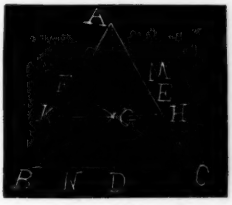
Three points are at random within a given triangle. Find the chance that they will all lie on one side of some one line that can be drawn through the center of gravity of the triangle.

Solution by G. B. M. ZERR, A. M., Ph. D., The Temple College, Philadelphia, Pa., and J. R. HITT, Coronal Institute, San Marcos, Tex.

Project the given triangle into an equilateral triangle, side$=a$; let G be the center of gravity, CH or $CM=x$.

Then $BK=\frac{a(a-2x)}{2a-3x}$, $AK=\frac{a(a-x)}{2a-3x}$, $AH=a-x$.

Area $AKH=\frac{\triangle(a-x)^2}{a(2a-3x)}$, $BN=\frac{2ax-a^2}{3x-a}$,



$$CN=\frac{ax}{3x-a}.$$

Area $CMN=\frac{\triangle x^2}{a(3x-a)}$, area $ABNM=\triangle[1-\frac{x^2}{a(3x-a)}]$.

$$p=\Big[\int_0^{\frac{1}{2}a}\frac{(AKH)^3dx}{\triangle^3}+\int_{\frac{1}{2}a}^{a}\frac{(ABNM)^3dx}{\triangle^3}\Big]/\int_0^{\frac{1}{2}a}dx$$

$$=\frac{2}{a}\int_{\frac{1}{2}a}^{a}\Big[1-\frac{x^2}{a(3x-a)}\Big]^3dx+\frac{2}{a^4}\int_0^{\frac{1}{2}a}\frac{(a-x)^6dx}{(2a-3x)^3}$$

$$=\frac{2}{a}\int_{\frac{1}{2}a}^{a}\Big[1-\frac{x^2}{a(3x-a)}\Big]^3dx+\frac{2}{a^4}\int_{\frac{1}{2}a}^{a}\frac{x^6dx}{(3x-a)^3}$$

$$=\frac{2}{a}\int_{\frac{1}{2}a}^{a}dx-\frac{6}{a^2}\int_{\frac{1}{2}a}^{a}\frac{x^2dx}{3x-a}+\frac{6}{a^3}\int_{\frac{1}{2}a}^{a}\frac{x^4dx}{(3x-a)^2}$$

$$=\tfrac{1}{27}(\tfrac{23}{2}-\tfrac{20}{3}\log 2).$$

116. Proposed by the late ENOCH BEERY SEITZ.

The average area of the quadrilateral formed by joining four random points on the surface of a circle, radius a, is $4a^2/3\pi$.

Solution by G. B. M. ZERR, A.M., Ph.D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let a=radius of given circle, A=its area, $\triangle$=required average, $\triangle'$=the average area when the four points are taken on both the circle A and a concentric annulus B, $\triangle_1$=the average area when three points are taken on A and one on B.

Then $(\triangle'-\triangle)A=4B(\triangle_1-\triangle)$.

But $\triangle:\triangle'=A:A+B$.

$\therefore\ \triangle'=\frac{(A+B)\triangle}{A}$. $\quad\therefore\ \triangle=\frac{4}{5}\triangle_1$.

Let one point O be on the circumference of the circle, and the other points P, Q, R anywhere on its surface. Let $OP=x$, $OR=y$, OQ

$=z$, $\angle AOD=\theta$, $\angle COD=\phi$, $\angle BOD=\psi$. Then $OA=2a\sin\theta=x'$, $OC=2a\sin\phi=y'$, $OB=2a\sin\psi=z'$. Area $OPQR=\frac{1}{2}xz\sin(\theta-\psi)+\frac{1}{2}yz\sin(\psi-\phi)=u$.

$$\therefore\ \Delta_1=\frac{\int_0^{\pi}\int_0^{\theta}\int_{\phi}^{\theta}\int_0^{x'}\int_0^{y'}\int_0^{z'} uxyzd\theta d\phi d\psi dxdydz}{\int_0^{\pi}\int_0^{\theta}\int_{\phi}^{\theta}\int_0^{x'}\int_0^{y'}\int_0^{z'} xyzd\theta d\phi d\psi dxdydz}$$

$$\therefore\ \Delta=\tfrac{4}{5}\Delta_1=\frac{24}{5\pi^3a^6}\int_0^{\pi}\int_0^{\theta}\int_{\phi}^{\theta}\int_0^{x'}\int_0^{y'}\int_0^{z'} uxyzd\theta d\phi d\psi dxdydz$$

$$=\frac{32}{5\pi^3a^3}\int_0^{\pi}\int_0^{\theta}\int_{\phi}^{\theta}\int_0^{x'}\int_0^{y'}[x\sin(\theta-\psi)+y\sin(\psi-\phi)]xy\sin^3\psi d\theta d\phi d\psi dxdy$$

$$=\frac{64}{15\pi^3a}\int_0^{\pi}\int_0^{\theta}\int_{\phi}^{\theta}\int_0^{x'}[3x\sin(\theta-\psi)+4a\sin\phi\sin(\psi-\phi)]\sin^2\phi\sin^3\psi d\theta d\phi d\psi xdx$$

$$=\frac{512a^2}{15\pi^3}\int_0^{\pi}\int_0^{\theta}\int_{\phi}^{\theta}[\sin\theta\sin(\theta-\psi)+\sin\phi\sin(\psi-\phi)]\sin^2\theta\sin^2\phi\sin^3\psi d\theta d\phi d\psi$$

$$=\frac{64a^2}{15\pi^2}\int_0^{\pi}\int_0^{\theta}[3(\phi-\theta)(\sin\theta\cos\theta-\sin\phi\cos\phi)+3(\sin\theta\cos\theta-\sin\phi\cos\phi)^2$$

$$+2\sin(\theta-\phi)(\sin^3\theta\cos\phi-\cos\theta\sin^3\phi)]\sin^2\theta\sin^2\phi d\theta d\phi$$

$$=\frac{2a^2}{45\pi^3}\int_0^{\pi}(105\theta-72\theta^2\sin\theta\cos\theta+84\theta\sin^2\theta-120\theta\sin^4\theta-105\sin\theta\cos\theta-82\sin^3\theta\cos\theta$$

$$+32\sin^5\theta\cos\theta)\sin^2\theta d\theta=4a^2/3\pi.$$

MISCELLANEOUS.

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111. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics in The Temple College, Philadelphia, Pa.

Exhibit $\cos^3\theta\sin^3\theta\sin^2\phi\cos\phi$ as a series of harmonics.

Solution by the PROPESER.

$$f(\mu,\ \phi)=\sum_{m=0}^{m=\infty}[A_{0,m}P_m(\mu)+\sum_{n=1}^{n=m}(A_{n,m}\cos n\phi+B_{n,m}\sin n\phi)P_m^n(\mu)].$$

Now $f(\mu,\ \phi)=\mu^3\sqrt{[(1-\mu^2)^3]}\sin^2\phi\cos\phi=\frac{1}{4}\mu^3(1-\mu^2)^{\frac{3}{2}}(\cos\phi-\cos3\phi)$.

$$A_{0,m}=\frac{2m+1}{16\pi}\int_{-1}^{1}\mu^3(1-\mu^2)^{\frac{3}{2}}P_m(\mu)d\mu\int_0^{2\pi}(\cos\phi-\cos3\phi)=0.$$

$$B_{n,m}=\frac{2m+1}{8\pi}\cdot\frac{(m-n)!}{(m+n)!}\int_{-1}^{1}\mu^3(1-\mu^2)^{\frac{3}{2}}P_m{}^n(\mu)d\mu\int_0^{2\pi}(\cos\phi-\cos3\phi)$$

$$\times\sin n\phi d\phi=0.$$

$$A_{n,m}=\frac{2m+1}{8\pi}\cdot\frac{(m-n)!}{(m+n)!}\int_{-1}^{1}\mu^3(1-\mu^2)^{\frac{3}{2}}P_m{}^n(\mu)d\mu\int_0^{2\pi}\cos\phi\cos n\phi d\phi=0,$$

unless $n=1$.

If $n=1$, $\int_0^{2\pi}\cos\phi\cos n\phi d\phi=\int_0^{2\pi}\cos^2\phi d\phi=\pi$.

$$A_{1,m}=\frac{2m+1}{8}\cdot\frac{(m-1)!}{(m+1)!}\int_{-1}^{1}\mu^3(1-\mu^2)^2\frac{dP_m(\mu)}{d\mu}d\mu$$

$$=\frac{1}{2^m.m!}\cdot\frac{2m+1}{8}\cdot\frac{(m-1)!}{(m+1)!}\int_{-1}^{1}\mu^3(1-\mu^2)^2\frac{d^{m+1}(\mu^2-1)^m}{(d\mu)^{m+1}}d\mu.$$

$$\int_{-1}^{1}\mu^3(1-\mu^2)^2\frac{d^{m+1}(\mu^2-1)^m}{(d\mu)^{m+1}}d\mu=-5040\int_{-1}^{1}\frac{d^{m-6}(\mu^2-1)^m}{(d\mu)^{m-6}}d\mu$$

by repeated integration by parts, $=0$ if $m>6$,

$$=-5040\int_{-1}^{1}(\mu^2-1)^6d\mu=-\frac{491520}{143}, \text{ if } m=6.$$

$$\therefore\ A_{1,6}=-\frac{1}{2^6.6!}\cdot\frac{13}{8}\cdot\frac{5!}{7!}\cdot\frac{491520}{143}=-\frac{2}{693}.$$

Similarly, $A_{1,5}=0$, $A_{1,4}=\frac{1}{770}$, $A_{1,3}=0$, $A_{1,2}=\frac{1}{63}$.

For $\frac{1}{4}\sin^3\theta\cos^3\theta\cos3\phi=\frac{1}{4}\mu^3\sin^3\theta\cos3\phi$

$$\mu^3=\frac{1}{6.5.4}\frac{d^3(\mu^6)}{(d\mu)^3}.\qquad \mu^6=\frac{16}{231}P_6(\mu)+\frac{24}{77}P_4(\mu)+\frac{10}{21}P_2(\mu)+\frac{1}{7}P_0(\mu).$$

$$\frac{d^3(\mu^6)}{(d\mu)^3}=\frac{16}{231}\cdot\frac{d^3P_6(\mu)}{(d\mu)^3}+\frac{24}{77}\cdot\frac{d^3P_4(\mu)}{(d\mu)^3}.\qquad\therefore\ \mu^3=\frac{2}{3465}\cdot\frac{d^3P_6(\mu)}{(d\mu)^3}+\frac{1}{385}\cdot\frac{d^3P_4(\mu)}{(d\mu)^3}.$$

$$\therefore\ \tfrac{1}{4}\sin^3\theta\cos^3\theta\cos\phi=-[\tfrac{2}{693}P_6'(\mu)-\tfrac{1}{770}P_4'(\mu)-\tfrac{1}{63}P_2'(\mu)]\cos\phi.$$

$$\tfrac{1}{4}\sin^3\theta\cos^3\theta\cos3\phi=-[\tfrac{1}{6930}P_6^3(\mu)+\tfrac{1}{1540}P_4^3(\mu)]\cos3\phi.$$

Now $P_m{}^n(\mu)=\sin^n\theta\frac{d^nP_m(\mu)}{(d\mu)^n}=(1-\mu^2)^{\frac{1}{2}n}\frac{d^nP_m(\mu)}{(d\mu)^n}$.

$$P_6(\mu)=\tfrac{1}{16}(231\mu^6-315\mu^4+105\mu^2-5).$$

$$P_4(\mu)=\tfrac{1}{8}(35\mu^4-30\mu^2+3. \quad P_2(\mu)=\tfrac{1}{2}(3\mu^2-1).$$

$$\therefore \sin^3\theta\cos^3\theta\sin^2\phi\cos\phi=-\tfrac{1}{44}(11\mu^3-3\mu)(1-\mu^2)^{\frac{3}{2}}\cos 3\phi$$

$$-\tfrac{3}{44}\mu(1-\mu^2)^{\frac{3}{2}}\cos 3\phi-\tfrac{1}{132}(33\mu^5-30\mu^3+5\mu)\sqrt{(1-\mu^2)}\cos\phi$$

$$+\tfrac{1}{308}(7\mu^3-3\mu)\sqrt{(1-\mu^2)}\cos\phi+\tfrac{1}{21}\mu\sqrt{(1-\mu^2)}\cos\phi.$$

PROBLEMS FOR SOLUTION.

ARITHMETIC.

162. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

A trolley road is built between two towns, and it is found that the gross annual receipts amount to 20% of the original cost; the annual cost of repairs is 2% of the original cost; and the working expenses is $3000 in addition to 20% of the net receipts. After a year a second road is built at the same cost as the first and it is found that the gross receipts and working expenses per year are doubled, while the cost of repairs for the new road is 1% of cost. If the net receipts for both roads is $72,500, find the cost of each road, and the net receipts the first year.

163. Proposed by CHRISTIAN HORNUNG, A.M., Professor of Mathematics, Heidelberg University, Tiffin, O.

Three Dutchmen and their wives went to market to buy hogs. The names of the men were Hans, Klaus and Hendricks, and of the women, Gertrude, Anna and Katrine; but it was not known which was the wife of each man. They each bought as many hogs as each man or woman paid shillings for each hog, and each man spent three guineas more than his wife. Hendricks bought 23 hogs more than Gertrude, and Klaus bought 11 more than Katrine. What was the name of each man's wife?

ALGEBRA.

166. Proposed by MARCUS BAKER, U. S. Geological Survey, Washington, D. C.

Solve

$$ax+by=2zx....[1].$$
$$cy+dz=2xy....[2].$$
$$ez+fx=2yz....[3].$$

167. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

A weight of m pounds falls and is broken into n pieces after which it is found that all weights, in pounds, from 1 to m can be weighed. Find the weight of each piece. Apply when $m=121$, $n=5$.

CALCULUS.

156. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Find the volume common to the two solids $x^2+y^2+z^2=a^2$ and $xz^2=(a-x)(x^2+y^2)$.

157. Proposed by L. C. WALKER, A. M., Petaluma High School, Petaluma, Cal.

Two equal ellipses are tangent to each other at the vertices of the major axes. If one of them be rolled on the other, find (1) the equation and area of the curve described by the vertex, and (2) by the center.

MECHANICS.

146. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

A diffraction grating, with lines .05 mm. apart is held in front of a Bunsen's burner in which common salt is volatilized, and, when viewed through a telescope it is found that the angular distances of the first, second, third, fourth, fifth, and sixth bright bands from the central one are respectively 41′, 1°21′, 2°2′, 2°42′, 3°23′ and 4°3′. Required the wave length of sodium light.

147. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

A particle mass m is attached to one end of a string, the other end of which is fixed. It is projected horizontally with such a velocity that it would rise to a position in which the string would be horizontal. But on its upward path it meets an inelastic particle mass m' and the height to which it rises is diminished by $1/p$th of what it would have risen. Find m', and the tensions of the string just after collision and at the greatest height of the particle.

DIOPHANTINE ANALYSIS.

106. Proposed by L. C. WALKER, A. M., Petaluma High School, Petaluma, Cal.

There is a series of rational triangles whose sides have a common difference of unity. Calling the one whose sides are 3, 4, 5 the first triangle, find the sides of the next five triangles, and a general expression for the sides of the nth triangle.

107. Proposed by L. C. WALKER, A. M., Petaluma High School, Petaluma, Cal.

Required the least three positive integral numbers such that the sum of all three of them, and the sum of every two of them shall be a square number.

NOTE.—Problem 105, Diophantine Analysis, May number, should read as follows:

Prove that every factor of $a^{2m}+b^{2m}$ is of the form $1 \pmod{2m+1}$ where a and b are prime to each other.

AVERAGE AND PROBABILITY.

131. Proposed by LON C. WALKER, A. M., Petaluma High School, Petaluma, Cal.

A sphere is described with its center within a given sphere, and its surface intersecting the surface of the given sphere. The average volume common to both spheres is 10/21 of the volume of the given sphere.

132. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

n points are taken at random on the circumference of a given circle. Prove that the chance of the center of the circle lying within the polygon formed by joining these points is $1-(1/2^{n-2})$.

MISCELLANEOUS.

128. Proposed by J. E. SANDERS, Hackney, Ohio.

The sides of a trapezium are $a=29$, $b=32$, $c=40$, and $d=36$. If c is opposite a, and the diagonals equal, what is the length of either diagonal?

129. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

How high above the surface of the earth must an observer be elevated at the latitude $\phi(=39^\circ\ 19')$, the declination of the sun being $\delta(=23^\circ\ 27')$, in order to see the sun at midnight?

NOTES.

Professor A. G. Greenhill was awarded by the London Mathematical Society its De Morgan Medal for 1902.

Professor W. H. Metzler, of Syracuse University, has been made Fellow of the Royal Society of Edinburgh.

Professor L. L. Locke has been elected Professor of Mathematics in Adelphi College, Brooklyn, New York.

Professor I. L. Fuchs, Professor of Mathematics in the University of Berlin since 1884, and of late editor of *Crelle's Journal*, died April 26th, at the age of sixty-eight years.

Dr. Charles W. M. Black, Instructor in Mathematics in the University of Oregon, and a contributor to the MONTHLY during the first two or three years of its publication, died August 11, at La Grande, Oregon.

Professors Ormond Stone, of the University of Virginia, E. H. Moore, of the University of Chicago, and Frank Morley, of Johns Hopkins University, have been appointed by the executive committee of the Carnegie Institution, as advisors in relation to original research in mathematics.

On July 6th occurred the death of William Lee Harvey, of Portland, Me. Mr. Harvey was born at Maxfield, Me., November 18, 1825. He was born and raised on a farm in the backwoods of Maine, and in his early years had only the advantages of the district schools of that day. He managed to spend a few terms in an academy and thus prepared himself for teaching in the common schools. While in school he acquired a taste for mathematics, and studied and

pretty thoroughly mastered algebra, geometry, trigonometry, and the calculus, without a teacher, using as texts Bonnycastle, Hutton, Young, and Peirce. As a problem-solver, Mr. Harvey ranked high, as many of his solutions of difficult problems proposed in the various mathematical journals of the country will show. He has in his library some rare works on mathematics, all of which are offered for sale. Persons interested in securing rare mathematical works should write to Mrs. Harvey, Portland, Me.

Since the last issue of the MONTHLY one of its loyal friends and supporters has been removed from among the great body of American mathematicians. On July 8th, occurred the death of Professor John D. Runkle, Professor of Mathematics in the Massachusetts Institute of Technology. Professor Runkle was a substantial friend of the MONTHLY from the beginning, and we shall greatly miss him. In the last letter we received from him, among other things, he said, "Please find enclosed my check for $5 in payment of my subscription to the MONTHLY for the current year. I am glad to make this small voluntary contribution to help you in your difficult work, for I know, by experience, how hard it is to maintain the publication of a mathematical journal in this country. You are doing a good work." Professor Runkle's experience with mathematical journalism was with the *Mathematical Monthly* which was founded by him in 1858 and which he edited and published for three years, at the end of which time it was discontinued for lack of proper support. We hope in a future number to publish a biographical sketch of Professor Runkle.

BOOKS AND PERIODICALS.

The Foundations of Geometry. By David Hilbert, Ph. D., Professor of Mathematics, University of Göttingen. Authorized Translation by E. J. Townsend, Ph. D., University of Illinois. 8vo. Cloth, 132 pages. Price, $1.00 Chicago: The Open Court Publishing Co.

In this work Professor Hilbert attempts "to choose for geometry a *simple* and *complete* set of independent axioms and to deduce from these the most important geometrical theorems in such a manner as to bring out as clearly as possible the significance of the different groups of axioms and the scope of the conclusions to be derived from the individual axioms."

He begins by considering *three* systems of things which he calls *points*, *straight lines*, and *planes*, and sets up a system of axioms connecting these things in their mutual relations. The axioms he arranges in five groups. These are I. 1—7. Axioms of *connection*; II. 1—5. Axioms of *order*; III. Axioms of *parallels* (Euclid's axiom); IV. 1—6. Axioms of *congruence*; V. Axiom of *continuity* (Archimedes' axiom). He then discusses the relations of these axioms to one another and also the bearing of each upon the logical development of Euclidean Geometry. With these axioms, Professor Hilbert arrives at many important results. The dependence of some of Professor Hilbert's axioms on others has been referred to in a previous issue of the MONTHLY. The work is not only of great mathematical value, but it is also of great importance from the pedagogical standpoint to the teacher of mathematics. Every teacher of geometry should carefully study this book. B. F. F.

A Philosophical Essay on Probabilities. By Pierre Simon, Marquis de Laplace. Translated from the Sixth French Edition by Frederick Wilson Truscott, Ph. D., Professor of Germanic Languages in the West Virginia University, and Frederick Lincoln Emory, M. E., Professor of Mechanics and Applied Mathematics in the West Virginia University. 8vo. Cloth, iv+196 pages. Price, $2.00 New York: John Wiley & Sons.

In this essay are set forth in a very simple way and without the aid of analysis, the principles and general results of the theory of probability. As the principles of probability obtain in the most important problems of life, this book will be of interest to the student of the humanities as well as to the student of mathematics. This is certainly the most elementary exposition of the theory that has ever been written. The fact that Laplace is the great authority on the subject from the analytical point of view gives this essay double value. B. F. F.

An Introduction to Celestial Mechanics. By F. R. Moulton, Ph. D., Instructor in Astronomy in the University of Chicago. Large 8vo. Cloth, xv+384 pages. Price, $3.50. New York: The Macmillan Co.

The author says, in the preface, "the aim has been to prepare such a book that one who has had the necessary mathematical training may obtain from it in a relatively short time and by the easiest steps a sufficiently broad and just view of the whole subject to enable him to stop with much of real value in his possession, or to pursue to the best advantage any particular portion he may choose." This is certainly an ambitious aim, but we believe that the author has well accomplished his aim. The work begins with a treatment of simple mechanical principles and solutions of problems in which the principles are used. From these elementary considerations the student is *gradually* led to the consideration of the most difficult problems in Astronomy. Logical sequence of the various subjects and the relative prominence which their scientific and educational importance deserve, have been maintained throughout the book. Many problems have been solved to illustrate the various principles established. The Introduction to the Problem of Three Bodies is very explicit. The work most successfully bridges the gap between the somewhat Popular or Descriptive treatment of Astronomy on the one hand, and the Exhaustive Mathematical treatment on the other. The work is desined to do great good from both the mathematical and astronomical standpoint. Both the author and the publishers are to be congratulated on presenting this valuable work to the educational public. B. F. F.

Annals of Mathematics. Published Quarterly under the Auspices of Harvard University. Price, $2.00 per year.

No. 4, of Vol. 3 (July number) contains the following articles: Note on a Twisted Curve with an Involution of Pairs of Points in a Plane, by Prof. H. S. White; On Some Curves Connected with a System of Similar Conics, by Prof. R. E. Allardice; Note on Multiply Perfect Numbers, by Jacob Westlund; A Mechanical Construction of Confocal Conics, by Wm. R. Ransom; On Sophus Lie's Representation of Imaginaries in Plane Geometry, by Prof. Percy F. Smith; Note on the Group of Isomorphism of a Group of Order p^m, by Dr. G. A. Miller. B. F. F.

The American Journal of Mathematics. Published Quarterly under the Auspices of Johns Hopkins University. Price, $5.00 per year.

No. 3, Vol. XXIV, contains the following articles: Die Typen der linearen Complexe elliptische Curvenim R^r, Von S. Kantor; Generalization of the Differentiation Process, by Robert E. Moritz; Simple Pairs of Parallel W-Surfaces, by Henry D. Thompson. B. F. F.

Arithmetica Razionale ad Uso dei Ginnasi. Autore Roldolfo Bettazzi, Prof. Nella R. Accademia Militare e Nel R. Liceo Cavour. Prezzo, L. 2,00 Torino.

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THE EXPRESSION OF THE nth POWER OF A NUMBER IN TERMS OF THE nth POWERS OF OTHER NUMBERS, n BEING ANY INTEGER; AND THE DEDUCTION OF SOME INTERESTING PROPERTIES OF PRIME NUMBERS.

By J. W. NICHOLSON, A. M., LL. D., Professor of Mathematics in the Louisiana State University.

[Concluded from August-September Number.]

Making $m=7$, we have

$$8^n=6^{n+1}-\tfrac{7}{2}.4^{n+1}+7.2^{n+1}=6.6^n-14.4^n+14.2^n....(37),$$

where $n=5$, 3, or 1.

When m is even and n is odd, or vice versa, (34) becomes

$$(m+1)^n=(m+1)(m-1)^n-\frac{(m+1)(m)}{2}(m-3)^n$$

$$+\frac{(m+1)\,(m)(m-1)}{6}(m-5)^5-\text{ etc.,}\ ....(38).$$

COR. V. In (21) m may be made equal to n provided that $n!\,a_1a_2....a_n$ be added to the second member.

Suppose these changes to be made and denote the result by....(39).

EXAMPLES.

Making $n=3$, in (39), we have

$$(c+a_1+a_2+a_3)^3=(c+a_1+a_2)^3+(c+a_1+a_3)^3+(c+a_2+a_3)^3 \\ -(c+a_1)^3-(c+a_2)^3-(c+a_3)^3+c^3+1.2.3.a_1a_2a_3....(40).$$

In (39), by making $a_1=a_2=....a_n$, and transposing, we have

$$n!\ a_1{}^n=(c+na_1)^n-n[c+(n-1)a_1]^n+\frac{n(n-1)}{2}[c+(n-2)a_1]^n-.... \\ +(-1)^nc^n....(41).$$

Making $a_1=1$, we have

$$n!=(c+n)^n-n(c+n-1)^n+....-(-1)^nn(c+1)^n+(-1)^nc^n....(42).$$

Now making $c=0$, we have

$$n!=n^n-n(n-1)^n+\frac{n(n-1)}{2}(n-2)^n-....-(-1)^nn(1)^n....(43).$$

Thus, for $n=4$, we have

$$1.2.3.4=4^4-4.3^4+6.2^4-4.1^4....(44).$$

For $n=6$ and $c=5$, (42) becomes

$$1.2.3.4.5.6=11^6-6.10^6+15.9^6-20.8^6+15.7^6-6.6^6+5^6....(45).$$

SOME INTERESTING PROPERTIES OF PRIME NUMBERS.

In the following $m+1$ is supposed to be a prime number, and m', m'', etc. represent integers.

Since each of the binomial coefficients in (25), $+1$ or -1, is divisible by $m+1$), (25) may be written

$$(c+ma_1)^n+[c+(m-1)a_1]^n+....(c+a_1)^n+c^n=m'(m+1)....(46).$$

That is, the first member, in which c and a_1 may be any integers and n any integer less than m, is exactly divisible by the prime number $m+1$.

Making $c=0$ and $a_1=1$, (46) becomes

$$(m)^n+(m-1)^n+(m-2)^n+....2^n+1=m''(m+1)....(47).$$

In (46), writing $2m$ for m, making $a_1=1$, $c=-m$, and supposing n an even number, we have